NOTES ON THE BIRKHOFF CONSTRUCTION OF FREE ALGEBRAS

RALPH FREESE

This is an explanation of the Birkhoff construction of the free algebra [3]. It borrows heavily from Berman's survey article [2] but uses slight different notation.

Let X be a set an $\mathbf{A}$ and algebra. A map $v : X \to A$ (so $v \in A^X$) is called a *valuation* or an *X–labelling*. Note: not all elements of A get labels and some can get more than one label. Let $\mathbf{A}(v)$ denote the subalgebra generated by $v(X)$. Let $u : X \to B$ be a labelling of an algebra $\mathbf{B}$. A homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ *respects the labellings* if $\varphi(v(x)) = u(x)$, for $x \in X$. This can be visualized with the following commutative diagram. (There really should just be one X; but diagonal arrows require a picture environment.)

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ \mathbf{A}(v) & \xrightarrow[\varphi]{} & \mathbf{B}(u) \end{array}$$

Let $\mathcal{K}$ be a set of algebras of the same similarity type and let $\mathcal{V} = \boldsymbol{HSP}\mathcal{K}$ be the variety generated by $\mathcal{K}$. First consider the case $\mathcal{K} = \{\mathbf{A}\}$ consists of a single algebra. We form the direct product of the algebras $\mathbf{A}(v)$, $v \in A^X$. This is an X–labelled algebra under the labelling

$$x \mapsto \bar{x} \in \prod_{v \in A^X} \mathbf{A}(v) \quad \text{where} \quad \bar{x}_v = v(x)$$

so $\bar{x}$ is the vector whose v^{th} component is $v(x)$. Viewing elements of a product as funtions on the index set, we can write this as $\bar{x}(v) = v(x)$. Let $\mathbf{F}$ be the subalgebra of $\prod_{v \in A^X} \mathbf{A}(v)$ generated by $\{\bar{x} : x \in X\}$. $\mathbf{F}$ has the universal mapping property over X for $\mathcal{K}$: if $u : X \to A$ then the restriction of the projection homomorphism π_u to F followed by the embedding of $\mathbf{A}(u)$ into $\mathbf{A}$ is a homomorphsim of $\mathbf{F}$ to $\mathbf{A}$ extending u.

Now if $\mathbf{U}$ has the universal mapping property for $\mathcal{K}$, it has the universal mapping property for $\boldsymbol{H}\mathcal{K}$, $\boldsymbol{S}\mathcal{K}$ and $\boldsymbol{P}\mathcal{K}$. Thus $\mathbf{F}$ has the universal mapping property for $\mathcal{V}$ and, since $\mathbf{F} \in \mathcal{V}$, it is the free algebra $\mathbf{F}_{\mathcal{V}}(X)$.

In the case $\mathcal{K}$ is a set of algebras, we form the direct product

$$\prod_{\mathbf{A} \in \mathcal{K}} \prod_{v \in A^X} \mathbf{A}(v)$$

As before X is naturally embedded into this product and the subalgebra generated by the image of X is the Birkhoff construction of the free algebra $\mathbf{F}_{\mathcal{V}}(X)$.

Two corollaries of this construction:

Corollary 1. *$\mathbf{F}_{\mathcal{V}}(X)$ is a subdirect product of subalgebras of members of $\mathcal{K}$. In particular $\mathbf{F}_{\mathcal{V}}(X) \in \boldsymbol{SP}\mathcal{K}$.*

Corollary 2. *$\mathbf{F}_{\mathbf{V}(\mathbf{A})}(X)$ is the subalgebra of $\mathbf{A}^{A^X}$ generated by $\{\bar{x} : x \in X\}$. In particular, if $|X| = k$ and $|A| = n$, then*

$$|\mathbf{F}_{\mathbf{V}(\mathbf{A})}(X)| \leq n^{n^k}.$$

1. Thinning the coordinates

Under the Birkhoff representation of the free algebra each element is a vector of length $|A|^{|X|}$, which can be rather large. For Lyndon's 7 element nonfinitely based groupoid this means every element of the free algebra on 6 generators, which turns out to have only $1,957$ elements, is a vector of length $7^6 = 117,649$ so every multiplication in $\mathbf{F}_{\mathbf{V}(\mathbf{A})}(X)$ becomes $117,649$ multiplications in $\mathbf{A}$. Often not all of the coordinates are necessary.

Let π_v be the projection homomorphism of the Birkhoff represenation of $\mathbf{F}_{\mathbf{V}(\mathbf{A})}(X)$ onto $\mathbf{A}(v)$ and let $\eta_v \in \mathbf{Con}\,(\mathbf{F}_{\mathbf{V}(\mathbf{A})}(X))$ be its kernel. Also let π'_v be the projection homomorphism onto all coordinates except the v^{th} coordinate. And let η'_v be its kernel. Of course $\bigwedge_{v \in A^X} \eta_v = 0$ in $\mathbf{Con}\,(\mathbf{F}_{\mathbf{V}(\mathbf{A})}(X))$. If $I \subseteq A^X$ such that $\bigwedge_{v \in I} \eta_v = 0$ then the natural map that maps each vector of $\mathbf{F}_{\mathbf{V}(\mathbf{A})}(X)$ to just the I^{th} coordinates is an isomorphism. In fact if two vectors in the Birkhoff representation agree on I, they are equal.

While in most cases it would be difficult to find a minimal set I such that $\bigwedge_{v \in I} \eta_v = 0$, we can at least eliminate some of the v's. If $\eta_v \leq \eta_u$ then clearly

$$\eta'_u = \bigwedge_{\substack{v \in A^X \\ v \neq u}} \eta_v = 0$$

and so u can be eliminated. The following lemma helps.

Lemma 3. *$\eta_v \leq \eta_u$ if and only if there is a homomorphism $\varphi : \mathbf{A}(v) \to \mathbf{A}(u)$ respecting the labelling.*

Proof. First suppose there is a label-respecting homomorphism $\varphi : \mathbf{A}(v) \to \mathbf{A}(u)$ and suppose $\bar{a}\ \eta_v\ \bar{b}$ for some $\bar{a}, \bar{b} \in \mathbf{F}_{\mathbf{V}(\mathbf{A})}(X)$. Since $\mathbf{F} := \mathbf{F}_{\mathbf{V}(\mathbf{A})}(X)$ is the subalgebra of $\mathbf{A}^{A^X}$ generated by the $\bar{x}$'s, there are terms t and s such that

$$\bar{a} = t^{\mathbf{F}}(\bar{x}_1, \ldots, \bar{x}_k) \quad \text{and} \quad \bar{b} = s^{\mathbf{F}}(\bar{x}_1, \ldots, \bar{x}_k); \tag{1}$$

see Theorem 10.3(c) of [4]. Applying π_v to these we get

$$a_v = \pi_v(\bar{a}) = t^{\mathbf{A}(v)}(v(x_1), \dots, v(x_k))$$
$$b_v = \pi_v(\bar{b}) = s^{\mathbf{A}(v)}(v(x_1), \dots, v(x_k))$$

Since $a_v = b_v$,

$$t^{\mathbf{A}(v)}(v(x_1), \dots, v(x_k)) = s^{\mathbf{A}(v)}(v(x_1), \dots, v(x_k)).$$

Applying φ to this, and using that $\varphi(v(x)) = u(x)$ gives

$$t^{\mathbf{A}(u)}(u(x_1), \dots, u(x_k)) = s^{\mathbf{A}(u)}(u(x_1), \dots, u(x_k)).$$

Using this and (1)

$$\begin{aligned} a_u = \pi_u(\bar{a}) &= \pi_u(t^{\mathbf{F}}(\bar{x}_1, \dots, \bar{x}_k) \\ &= t^{\mathbf{A}(u)}(u(x_1), \dots, u(x_k)) \\ &= s^{\mathbf{A}(u)}(u(x_1), \dots, u(x_k)) \\ &= \pi_u(s^{\mathbf{F}}(\bar{x}_1, \dots, \bar{x}_k) = \pi_u(\bar{b}) = b_u \end{aligned}$$

showing $\bar{a}\ \eta_u\ \bar{b}$. This proves one direction. For the other, we need to assume $\eta_v \leq \eta_u$ and show that the map $v(x) \mapsto u(x)$ extends to a homomorphism $\varphi : \mathbf{A}(v) \to \mathbf{A}(u)$. We leave this as an exercise. □

In the Lyndon groupoid we can use this thinning to reduced the number of coordinates from $117,649$ down to $22,526$.

We can extend Lemma 3 to test if $\eta_u \wedge \eta_v \leq \eta_w$. This will be the case if and only if there is a homomorphsim respecting X from the subagebra of $\mathbf{A}(u) \times \mathbf{A}(v)$ generated by $\{(u(x), v(x)) : x \in X\}$ onto $\mathbf{A}(w)$. However, this is usually too expensive since it involves looking at all 3 element subsets of A^X.

Exercises

1. Let $\mathbf{N}_5$ be the 5 element nonmodular lattice. Using the thinning process described above show that when $|X| = 3$, the 125 coordinates of $\mathbf{F}_{\mathbf{N}_5}(X) \leq \mathbf{N}_5^{N_5^X}$ are thinned down to 18 coordinates. And find the sizes of the $\mathbf{N}_5(v)$ that remain.

2. Show that when $|X| = 4$ the proceedure thins the 625 coordinates down to 132 and find the sizes of the $\mathbf{N}_5(v)$ that remain.

3. Show that when $|X| \geq 5$ the $\mathbf{N}_5(v)$'s that remain after thinning all have size 4 or 5.

Subdirect Decompositions. While it is difficult to find all subdirectly irreducible homomorphic images of an algebra $\mathbf{A}$, one can find a subdirect decomposition into subdirectly irreducibles in polynomial time: First find the principal congruence of $\mathbf{A}$ (this can be done in polynomial time by [5]). Then find the subset of join irreducibles and the atoms. For each atom find a meet irreducible congruence not above it by joining principles.

We can apply this to each of our $\mathbf{A}(v)$'s. In $\mathbf{Con}\ (\mathbf{F}_{\mathbf{V}(\mathbf{A})}(X))$ we are replacing η_v with a set $\{\psi_{v,i}\}$ of meet irreducibles which meet to η_v. This seems counter-productive since it actually makes the vectors longer but the thining process applied to the $\psi_{v,i}$ is usually able to produce a smaller set of coordinates. In the Lyndon algebra it reduces the number of coordinates down to $11,582$. This process works particularly well for congruence distributive algebras where it produces the unique minimal subdirect representation into subdirectly irreducibles. The next example illustrates this.

Example. Consider $\mathbf{F}_{\mathbf{V}(\mathbf{A})}(x, y, z)$, where $\mathbf{A} = \mathbf{N}_5$ is the five element nonmodular lattice. $\mathbf{N}_5$ has several sublattices isomorphic to $\mathbf{3}$, the three element chain. If u and $v \in A^X$, $X = \{x, y, z\}$, are labellings such that $z < y < x$ in both, then there is an isomorphism $\mathbf{A}(u) \to \mathbf{A}(v)$ respecting the labelling. So $\eta_u = \eta_v$ and the thinning can eliminate most of this projections. But this still leaves 6 projections onto the three element chain corresponding the the 6 orderings of X. Since there is no homomorphism of $\mathbf{N}_5$ or of $\mathbf{2} \times \mathbf{2}$ onto $\mathbf{3}$, all 6 of these coordinates will be retained in our thinning process. In fact the thinning process leaves 18 of the 125 projections.

But it is easy to see that if $\mathbf{A}(v)$ is the two element lattice, then there is a labelling u such that $\mathbf{A}(u) \cong \mathbf{N}_5$ and there is a homomorphism $\mathbf{A}(u) \to \mathbf{A}(v)$ respecting the labelling. These arguments show that using the subdirect decomposition method described above, all projections except the 6 where $\mathbf{A}(v) \cong \mathbf{N}_5$ can be eliminated. You can play with this using the Univesal Algebra Calculator (at `www.uacalc.org`). Choose the built-in algebra `n5`, and form the free algebra on 3 generators, one time choosing “thin coords” and one time choosing “decompose and thin”. We summarize our observation about $\mathbf{N}_5$.

Theorem 4. *For* $|X| \geq 3$, $\mathbf{F}_{\mathbf{V}(\mathbf{N}_5)}(X)$ *is a subdirect product of copies of* $\mathbf{N}_5$.

In the case $|X| = 4$, $\mathbf{F}_{\mathbf{V}(\mathbf{N}_5)}(X)$ is a subdirect product of 84 copies of $\mathbf{N}_5$. The size of the free algebra is $540,792,672$, as Berman and Wolk showed [1]. This is too big for the Calculator to construct. But for $\mathbf{M}_3$ it can construct the free algebra on 4 generators. It is a subdirect product of 14 copies of $\mathbf{M}_3$ and 14 copies of $\mathbf{2}$ and has $19,982$ elements.

2. Birkhoff Basis

In testing if there is a homomorphism from $\mathbf{A}(v)$ onto $\mathbf{A}(u)$ respecting the labelling we first check if the map $\varphi : v(x) \mapsto u(x)$ is well defined. (This can fail if, say, $v(x) = v(y)$ but $u(x) \neq u(y)$.) If φ is well defined, we try to extend it to all of $\mathbf{A}(v)$. We build $\mathbf{A}(v) \leq \mathbf{A}$ by closing the image of v under the operations in the usual way and try to extend φ as we go.

So if a and b have been generated and f is a binary operation, we form $e = f(a, b)$. If this element is not in the closure we have so far, we add it and we extend φ by setting $\varphi(e) = f(\varphi(a), \varphi(b))$. On the other hand if e is already in the closure so $\varphi(e)$ is already defined, we check if the existing $\varphi(e)$ is equal to $f(\varphi(a), \varphi(b))$. If this fails, we stop: there is no homomorphism $\mathbf{A}(v) \to \mathbf{A}(u)$ respecting the labelling. Otherwise we continue.

Let $\mathcal{V} = \boldsymbol{V}(\mathbf{A})$ and let $\mathbf{F} := \mathbf{F}_{\mathcal{V}}(X)$ be the Birkhoff construction as the subalgebra of $\prod_{v \in A^X} \mathbf{A}(v)$ generated by $\{\bar{x} : x \in X\}$, $\bar{x}_v = v(x)$. Using ideas similar to the last paragraph, when closing $\{\bar{x} : x \in X\}$ we can associate to each element a of the closure a term t_a in the variables X such that $t_a^{\mathbf{F}} = a$. Namely the term of $\bar{x}$ is x and if c first appears as $f(a, b)$ then $t_c = f(t_a, t_b)$.

Now suppose g is an r-ary operation symbol, $a_1, \ldots, a_r \in \mathbf{F}$ and $b = g^{\mathbf{F}}(a_1, \ldots, a_r)$. Then the equation

$$t_b \approx g(t_{a_1}, \ldots, t_{a_r}) \tag{2}$$

is true in $\mathbf{A}$. Indeed, a substitution of the variables X into $\mathbf{A}$ is an element $v \in A^X$. If $x_1, \ldots x_k$ are the variables occuring in the equation, then by the way the terms t_a were defined, the relation

$$t_b^{\mathbf{F}}(x_1, \ldots, x_k) = g^{\mathbf{F}}(t_{a_1}^{\mathbf{F}}(x_1, \ldots, x_k), \ldots, t_{a_r}^{\mathbf{F}}(x_1, \ldots, x_k))$$

holds in $\mathbf{F}$. Now just apply π_v to show that (2) holds in $\mathbf{A}$ under this substitution.

The set of all equations of the form (2) for all basic operations g and all r-tuples $a_1, \ldots, a_r$, r the arity of g, is called the *Birkhoff basis* for the equations of $\mathbf{A}$ in the variables X. Recall that we have associated with each element a of $\mathbf{F}$ a term t_a) with variables in X. One can use the Birkhoff basis to transform an arbitrary term in X, $t(x_1, \ldots, x_k)$, $x_i \in X$ into the term t_a, where $a = t^{\mathbf{F}}(x_1, \ldots, x_k)$. Then, if s and t are terms in X, we can decide if $s \approx t$ holds in $\mathcal{V}$ by transforming both s and t as above and seeing if the results are equal. The details are left as an exercise.

By Corollary 2 if $\mathbf{A}$ is finite then the Birkhoff basis is finite. So we have the following corollary.

Corollary 5. *If $\mathbf{A}$ is a finite algebra, then the k-variable equations of $\mathbf{A}$ are finitely based, for each finite k.*

As we mentioned before, there are finite algebras with only finitely many basic operations that do not have a finite equational basis for their identities.

Finally we mention that all of the above could be done for a set $\mathcal{K}$ of algebras in place of the single algebra $\mathbf{A}$.

References

[1] J. Berman and B. Wolk, *Free lattices in some small varieties*, Algebra Universalis **10** (1980), 269–289.

[2] Joel Berman, *The structure of free algebras*, Structural theory of automata, semigroups, and universal algebra, NATO Sci. Ser. II Math. Phys. Chem., vol. 207, Springer, Dordrecht, 2005, pp. 47–76.

[3] G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Phil. Soc. **31** (1935), 433–454.

[4] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, New York, 1981.

[5] Ralph Freese, *Computing congruences efficiently*, Algebra Universalis **59** (2009), 337–343, Online manuscript available at: http://www.math.hawaii.edu/∼ralph/papers.html.

(Ralph Freese) Department of Mathematics, University of Hawaii, Honolulu, Hawaii, 96822 USA

E-mail address, Ralph Freese: ralph@math.hawaii.edu